# A CHARACTERIZATION BASED ON PRODUCTS OF ORDER STATISTICS

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Abstract. A new characterization for power function distributions is obtained which is based on products of order statistics. This result may be considered as a generalization of some recent results for contractions. We note that in this new result the product consists of order statistics from independent samples. These type of results are related to some scheme of ranked set sampling, for example.

## 1. Introduction

In this paper, some distributional relations and characterizations related to products of order statistcis are investigated. Also, some new distributional relations involving products of order statistics, which may also lead to characterizations, are given. One important difference in the considered relations is that products of order statistics from independent samples are used.

Consider three independent random variables $X$, $Y$, and $U$, where $U$ has some known distribution. There are several recent results for relations of the type

$$X \stackrel{d}{=} YU \tag{1}$$

In its most basic form $U$ can be assumed to have a uniform distribution concentrated on (0,1). In this case, relation (1) is an example of a contraction. These type of relations have some applications like in economic modelling and reliability, for example. Some of the first results of this type were obtained, among others, by [Alamatsaz(1985)], [Kotz and Steutel(1988)], and [Alzaid and Al-Osh(1991)].

There are many interesting distributional relations and characterization results based on this type of relation. In this paper, we will consider relations where $X$ and $Y$ will be some order statistic.

We will write $X \sim Pow(\alpha)$ if $F_X(x) = x^{\alpha},\ \alpha > 0,\ x \in (0,1)$, and $Y \sim Par(\alpha)$, if $F_Y(y) = 1 - y^{-\alpha},\ \alpha > 0,\ y \in (1,\infty)$.

Let $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$ denote the order statistics for random variables $X_1, X_2, \ldots, X_n$. Similarly, let $X^*_{1:n}, X^*_{2:n}, \ldots, X^*_{n:n}$ denote the generalized (or $m$-generalized) order statistics for an underlying distribution function $F$. For more details about generalized order statistics, the reader is referred to [Kamps(1995)].

[Beutner and Kamps(2008)] considered characteriaztions of type (1), where $U \sim Pow(\alpha)$ or $U \sim Par(\alpha)$, and $X, Y$ are neighboring generalized order statistics. In particular, for $U \sim Pow(\alpha)$, they obtained characterizations based on the following relations:

$$X^*_{r:n} \stackrel{d}{=} X^*_{r+1:n} \cdot U,$$

$$X^*_{r:n} \stackrel{d}{=} X^*_{r:n-1} \cdot U, \tag{2}$$

$$X^*_{r:n-1} \stackrel{d}{=} X^*_{r+1:n} \cdot U.$$

The results in [Beutner and Kamps(2008)] generalize some other characterizations, given, for example, in [Oncel et al.(2005)], or [Wesołowski and Ahsanullah(2004)]. The main result of this paper is a characterization of type (1), which is similar to the second relation in (2). We note also that in this new result, instead of given $U$, we consider a product of two order statistics from an unknown distribution $F$.

## 2. Results

Let $X_1, X_2, \ldots, X_n$ be iid random variables with continuos distribution function $F(x)$. If $X_i \sim Pow(1)$, that is, uniformly distributed on $(0,1)$, then it can be shown that the following relation is true:

$$U_{1:n} \stackrel{d}{=} U_{1:1} U_{2:2} \cdots U_{n:n}$$

Here the $U_i$ denote the order statistics from a uniform distribution on $(0,1)$. In fact, the following result is true as well, which is given here without proof.

**Theorem 1.** *Let $X_1,\ X_2,\ ...,\ X_n,\ ...$ be iid absolutely continuous random varables with cdf $F(x),\ x \in (0,1)$. If $\{U_{i:i}\}$, $i \in \{1,\ 2,\ ...,\ n\}$ are independent, and*

$$X_{1:n} \stackrel{d}{=} U_{1:1} U_{2:2} \cdots U_{n:n},$$

*then $F(x) = x,\ x \in (0,1)$.*

**Remark 2.** *The proof of Theorem 1 can be shown directly with multiple integration but this result will follow directly from the next theorems.*

**Theorem 3.** *Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed (iid) random variables with cdf $Pow(\alpha)$, that is, $F(x) = x^\alpha$, on $(0,1)$. If $X_{k:n-1}$ and $X_{n:n}$ are independent, then*

$$X_{k:n} \stackrel{d}{=} X_{k:n-1} X_{n:n}. \tag{3}$$

*Proof.* Let $Y = X_{k:n-1} X_{n:n}$. Then

$$\begin{aligned} P(Y \le y) &= \int_0^1 F_{n:n}\left(\frac{y}{u}\right) f_{k:n-1}(u)\, du \\ &= F_{k:n-1}(y) + \int_y^1 F_{n:n}\left(\frac{y}{u}\right) f_{k:n-1}(u)\, du \end{aligned} \tag{4}$$

Hence, $X_{k:n} \stackrel{d}{=} X_{k:n-1} X_{n:n}$ implies that

$$F_{k:n}(x) = F_{k:n-1}(x) + \int_x^1 F_{n:n}\left(\frac{x}{u}\right) f_{k:n-1}(u)\, du. \tag{5}$$

Since for any $1 \le k \le n-1$, $n\left[F_{k:n} - F_{k:n-1}\right] f = F f_{k:n}$, (see, for example, [Wesołowski and Ahsanullah(2004)]), it follows that

$$\frac{F(x) f_{k:n}(x)}{n f(x)} = \int_x^1 F_{n:n}\left(\frac{x}{u}\right) f_{k:n-1}(u)\, du. \tag{6}$$

Evaluating the integral, using $X_i \sim Pow(\alpha)$,

$$\begin{aligned} \int_x^1 F_{n:n}\left(\frac{x}{u}\right) f_{k:n-1}(u)\, du &= k\binom{n-1}{k} \int_x^1 \left(\frac{x^\alpha}{u^\alpha}\right)^n (u^\alpha)^{k-1} (1-u^\alpha)^{n-k-1} \alpha u^{\alpha-1} du \\ &= k\binom{n-1}{k} x^{\alpha n} \int_x^1 (u^\alpha)^{-n+k-1} (1-u^\alpha)^{n-k-1} u^{\alpha-1} du \\ &= k\binom{n-1}{k} x^{\alpha n} \int_{x^\alpha}^1 t^{-n+k-1} (1-t)^{n-k-1} dt, \quad (t = u^\alpha) \\ &= k\binom{n-1}{k} x^{-\alpha n} \int_{x^\alpha}^1 (v-1)^{n-k-1} dv \\ &= k\binom{n-1}{k} x^{\alpha n} (x^{-\alpha} - 1)^{(n-k)} \end{aligned}$$

On the other hand, it follows that

$$\begin{aligned}\frac{F(x)f_{k:n}(x)}{nf(x)} &= k\binom{n-1}{k}[F(x)]^k[1-F(x)]^{n-k} \\ &= k\binom{n-1}{k}(x^\alpha)^k(1-x^\alpha)^{n-k}\end{aligned}$$

Hence, the proof is complete.

□

As an immediate consequence of this theorem, any order statistic $X_{k:n}$ from a power distribution can be expressed in terms of maximum order statistics:

**Corollary 4.** *Let $X_1, \ldots, X_n$ be iid with cdf $F(x) = x^\alpha$, $x \in (0,1)$, that is $X_i \sim Pow(\alpha)$. Then for any $1 \leq k \leq n-1$*

$$X_{k:n} \stackrel{d}{=} X_{k:k}X_{k+1:k+1}\cdots X_{n:n}.$$

**Remark 5.** *Note that for $k=1$ and $\alpha=1$, we obtain*

$$U_{1:n} \stackrel{d}{=} U_{1:1}U_{2:2}\cdots U_{n:n}.$$

To proof our main result, we will use the following lemma. We note that this lemma can be considered as another special variant of the Choquet-Deny Theorem. The proof of this lemma is almost the same as the proof of Theorem 1 in [Fosam and Shanbhag(1997)].

**Lemma 6.** *Let $H$ be a nonnegative function that is not identically equal to zero on $A=(0,1)$. Also, let $\{\mu_x : x \in A\}$ be a family of finite measures such that for each $x \in A$, $\mu(B_x) > 0$, where $B_x = (x,1)$. Then a continuous real-valued function $H$ on $A$ such that $H(x)$ has a limit as $x$ tends to $1-$, satisfies*

$$\int_x^1 \left[H(x) - H\left(\frac{x}{u}\right)\right]\mu_x(du) = 0,\ x \in (0,1), \tag{7}$$

*if and only if it is identically equal to a constant.*

**Theorem 7.** *Let $X_1, \ldots, X_n$ be iid with common continuous cdf $F$ and support $(0,1)$ such that $\sup F^{-1}(0) = 0$ and $\inf F^{-1}(1) = 1$. If for some fixed $1 \leq k \leq n-1$, $X_{k:n-1}$ is independent from $X_{n:n}$, and*

$$X_{k:n} \stackrel{d}{=} X_{k:n-1}X_{n:n},$$

*then $X_i \sim Pow(\alpha)$.*

*Proof.* $X_{k:n} \stackrel{d}{=} X_{k:n-1}X_{n:n}$ implies that

$$F_{k:n}(x) = F_{k:n-1}(x) + \int_x^1 F_{n:n}\left(\frac{x}{u}\right) f_{k:n-1}(u)du \tag{8}$$

Since $n\left[F_{k:n}(x) - F_{k:n-1}(x)\right] F(x) = nf(x)$, we have

$$F(x)f_{k:n}(x) = nf(x)\int_x^1 F_{n:n}\left(\frac{x}{u}\right) f_{k:n-1}(u)du \tag{9}$$

By differentiating (8) with respect to $x$, it follows that

$$f_{k:n}(x) = \int_x^1 f_{n:n}\left(\frac{x}{u}\right)\frac{1}{u} f_{k:n-1}(u)du \tag{10}$$

From (9) and (10), we obtain

$$nf(x)\int_x^1 F_{n:n}\left(\frac{x}{u}\right) f_{k:n-1}(u)du = F(x)\int_x^1 f_{n:n}\left(\frac{x}{u}\right)\frac{1}{u} f_{k:n-1}(u)du$$

or

$$\int_x^1 \left[nf(x)F_{n:n}\left(\frac{x}{u}\right) - F(x)f_{n:n}\left(\frac{x}{u}\right)\frac{1}{u}\right] f_{k:n-1}(u)du = 0,\ x \in (0,1)$$

This last equation can be written as

$$\int_x^1 F^n\left(\frac{x}{u}\right)\left[\frac{xf(x)}{F(x)} - \frac{\frac{x}{u}f\left(\frac{x}{u}\right)}{F\left(\frac{x}{u}\right)}\right] f_{k:n-1}(u)du = 0,\ x \in (0,1), \tag{11}$$

or, defining $H(x) = \frac{xf(x)}{F(x)}$,

$$\int_x^1 F^n\left(\frac{x}{u}\right)\left[H(x) - H\left(\frac{x}{u}\right)\right] f_{k:n-1}(u)du = 0,\ x \in (0,1). \tag{12}$$

Now, using Lemma 6 with $\mu_x(B) = \int_{B\cap B_x} F^n\left(\frac{x}{u}\right) f_{k:n-1}(u)du$, $B_x = (x,1)$, it follows that $H$ is constant on $(0,1)$;

$$H(x) = \frac{xf(x)}{F(x)} = k,\ x \in (0,1), \tag{13}$$

for some $k \in \mathbf{R}$. The solution of this separable differential equation with boundary conditions $F(0) = 0$ and $F(1) = 1$ implies that $F(x) = x^{\alpha},\ x \in (0,1)$.

□

## 3. Summary

A new characterization result based on products of order statistics from independent samples has been obtained. This characterization result is interesting because in some sense it is an extension of contraction type results studied in the literature. In addition, it may be related to some special scheme of ranked set sampling. Another interesting point of the main result is that as a by-product a representation theorem for order statistics is obtained. Actually, the $k$-th order statistic from a power function distribution can be expressed as a product of independent maximum order statistics as shown in Corollary 4.

We also note that the proof of the main result is obtained by applying a new variant (Lemma 6) of the Choquet-Deny Theorem (see, for example, [Fosam and Shanbhag(1997)]).

## References

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